

$\pi$ -METRIZABLE SPACES AND STRONGLY  $\pi$ -METRIZABLE SPACES

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**ABSTRACT.** A space  $X$  is said to be  $\pi$ -metrizable if it has a  $\sigma$ -discrete  $\pi$ -base. In this paper, we mainly give affirmative answers for two questions about  $\pi$ -metrizable spaces. The main results are that: (1) A space  $X$  is  $\pi$ -metrizable if and only if  $X$  has a  $\sigma$ -hereditarily closure-preserving  $\pi$ -base; (2)  $X$  is  $\pi$ -metrizable if and only if  $X$  is almost  $\sigma$ -paracompact and locally  $\pi$ -metrizable; (3) Open and closed maps preserve  $\pi$ -metrizability; (4)  $\pi$ -metrizability satisfies hereditarily closure-preserving regular closed sum theorems. Moreover, we define the notions of second-countable  $\pi$ -metrizable and strongly  $\pi$ -metrizable spaces, and study some related questions. Some questions about strongly  $\pi$ -metrizability are posed.

## 1. INTRODUCTION

$\pi$ -metrizable spaces were first studied by V. Ponomarev as a necessary conditions for being the absolute of a metrizable space [8]. In [6], D. Fearnley has constructed a Moore and  $\pi$ -metrizable space which cannot be densely embedded in any Moore space with the Baire property. In [10], D. Stover has proved that a space  $X$  is  $\pi$ -metrizable if and only if  $X$  has a  $\sigma$ -locally finite  $\pi$ -base. It is well known that a regular space is metrizable if and only if it has a  $\sigma$ -hereditarily closure-preserving base. Recently, C. Liu posed the following two questions in a private communication with the authors.

**Question 1.1.** *If  $X$  has a  $\sigma$ -hereditarily closure-preserving  $\pi$ -base, is  $X$   $\pi$ -metrizable?*

**Question 1.2.** *Is  $\pi$ -metrizability preserved by open and closed maps?*

Obviously, if the Question 1.1 is affirmative, then Question 1.2 is also affirmative.

In this paper, we shall give an affirmative answer for Questions 1.1 and 1.2, respectively. In fact, we prove that quasi-open and closed maps preserve  $\pi$ -metrizability. We also improve some results in [10]. Moreover, we define the notions of second-countable  $\pi$ -metrizable and strongly  $\pi$ -metrizable spaces, and study some related questions.

**Definition 1.3.** Let  $X$  be a space. A collection of nonempty open sets  $\mathcal{U}$  of  $X$  is called a  $\pi$ -base if for every nonempty open set  $O$ , there exists an  $U \in \mathcal{U}$  such that  $U \subset O$ . A space  $X$  is said to be  $\pi$ -metrizable if it has a  $\sigma$ -discrete  $\pi$ -base. A space  $X$  is called a *second-countable  $\pi$ -metrizable space* if  $X$  has a countable  $\pi$ -base.

Obviously, every second-countable  $\pi$ -metrizable space is  $\pi$ -metrizable.

**Definition 1.4.** Let  $f : X \rightarrow Y$  be a map.

- (1)  $f$  is a *compact map* if each  $f^{-1}(y)$  is compact in  $X$ ;
- (2)  $f$  is a *perfect map* if it is a closed and compact map;
- (3)  $f$  is a *quasi-open map* if  $\text{Int} f(U) \neq \emptyset$  for any non-empty open subset  $U$  of  $X$ ;

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- (4)  $f$  is called *at most  $k$ -to-one map* if  $|f^{-1}(y)| \leq k$  for every  $y \in Y$ , where  $k \in \mathbb{N}$ ;
- (5)  $f$  is an *irreducible map* if there does not exist a proper closed subset  $X'$  of  $X$  such that  $f(X') = Y$ .

**Definition 1.5.** [3] Let  $\mathcal{P}$  be a family of subsets of a space  $X$ .  $\mathcal{P}$  is *hereditarily closure-preserving* (abbrev. HCP) if, whenever a subset  $S(P) \subset P$  is chosen for each  $P \in \mathcal{P}$ , the family  $\{S(P) : P \in \mathcal{P}\}$  is closure-preserving.

**Definition 1.6.** [10] A space  $X$  is called *strongly  $d$ -separable* if there exists  $\{K_n : n \in \mathbb{N}\}$  such that each  $K_n$  is a closed discrete subset of  $X$  and  $\cup\{K_n : n \in \mathbb{N}\}$  is dense in  $X$ .

For a topological space  $X$ , let  $\mathcal{P}$  be a family of subsets of  $X$ , and let

$$I(X) = \{x : x \text{ is an isolated point of } X\},$$

$$(\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\} \text{ for each } x \in X.$$

However, we denote  $\mathcal{P}_x$  by a subfamily of  $(\mathcal{P})_x$  for each  $x \in X$ .

Throughout this paper, all spaces are assumed to be  $T_1$  and regular, all maps are continuous and onto. Denote the positive natural numbers by  $\mathbb{N}$ . We refer the reader to [5] for notations and terminology not explicitly given here.

## 2. $\pi$ -METRIZABLE SPACES

First, we give two technical lemmas in order to give an affirmative answer for Question 1.1.

**Lemma 2.1.** [3] Let  $\mathcal{P}$  be a HCP collection of open subsets of  $X$  and  $A \subset X$ . If  $x \in A^d$  and  $G$  is a  $G_\delta$ -set of  $X$  such that  $x \in G$  and  $G \cap (A - \{x\}) = \emptyset$ , then  $(\mathcal{P})_x$  is finite

**Lemma 2.2.** Let  $X$  have a  $\sigma$ -HCP  $\pi$ -base  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$ , where  $\mathcal{P}_n$  is HCP for each  $n \in \mathbb{N}$ . Then  $(\mathcal{P}_n)_x$  is finite for each  $x \in X \setminus I(X)$  and  $n \in \mathbb{N}$ .

*Proof.* Fix a point  $x \in X \setminus I(X)$ . For each  $n \in \mathbb{N}$  and  $P \in \mathcal{P}_n$ , we choose a point  $x_P \in P \setminus \{x\}$ . Let  $F_n = \{x_P : P \in \mathcal{P}_n\}$ . Then  $F_n$  is closed. Put  $A = \bigcup_{n \in \mathbb{N}} F_n$  and  $G = X - A$ . For each  $x \in U$  with  $U$  open in  $X$ , there exists a  $P \in \mathcal{P}$  such that  $P \subset U$ , and hence  $x_P \in U \cap (A - \{x\}) \neq \emptyset$ . Therefore,  $x \in A^d \cap G$ . Obviously,  $G$  is a  $G_\delta$ -set and  $G \cap (A - \{x\}) = \emptyset$ . Hence  $(\mathcal{P}_n)_x$  is finite by Lemma 2.1.  $\square$

A collection of sets  $\mathcal{U}$  in a space  $X$  each with nonempty interior is called a  $\pi_*$ -base [10] if for each open set  $O$  there is an  $U \in \mathcal{U}$  such that  $U \subset O$ .

**Theorem 2.3.** For a topological space  $X$ , the following are equivalent:

- (1)  $X$  is a  $\pi$ -metrizable space;
- (2)  $X$  has a  $\sigma$ -HCP  $\pi$ -base;
- (3)  $X$  has a  $\sigma$ -locally finite  $\pi$ -base.

*Proof.* (1) $\Rightarrow$ (2) is obvious. (3) $\Rightarrow$ (1) by [10, Theorem 2.2]. Hence we only need to prove (2) $\Rightarrow$ (3).

Let  $\mathcal{P} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n$  be a  $\sigma$ -HCP  $\pi$ -base of  $X$ , where each  $\mathcal{P}_n$  is HCP. By the regularity, for each  $P \in \mathcal{P}$ , there is a nonempty closed subset  $B_P$  in  $X$  such that  $B_P \subset P$ , and if  $P \not\subset I(X)$  then  $\text{int}(B_P) \cap (X \setminus I(X)) \neq \emptyset$ . Let  $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$ , where  $\mathcal{B}_n = \{B_P : P \in \mathcal{P}_n\}$  for each  $n \in \mathbb{N}$ . It is easy to see that  $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$  is a  $\sigma$ -HCP  $\pi_*$ -base of  $X$ . For each  $n \in \mathbb{N}$ , let  $X(n) = \{x \in X : \mathcal{B}_n \text{ is locally finite at point } x\}$ .

Claim: For each  $n \in \mathbb{N}$ ,  $X \setminus I(X) \subset X(n)$ .

Indeed, put  $x \in X \setminus I(X)$ . It follows from Lemma 2.2 that  $(\mathcal{P}_n)_x$  is finite, and hence  $(\mathcal{B}_n)_x$  is also finite. Therefore,  $\cup(\mathcal{B}_n \setminus (\mathcal{B}_n)_x)$  is closed and does not contain  $x$ , and hence  $X \setminus (\cup(\mathcal{B}_n \setminus (\mathcal{B}_n)_x))$  is an open neighborhood of  $x$  and at most intersects finitely many elements of  $\mathcal{B}_n$ . So,  $x \in X(n)$ .

It is obvious that  $X(n)$  is open for each  $n \in \mathbb{N}$ . Let  $\mathcal{P}'_n = \{\text{int}(B) \cap X(n) : B \in \mathcal{B}_n\}$ . Then  $\mathcal{P}'_n$  is a locally finite collection of open subsets of  $X$  for each  $n \in \mathbb{N}$ . Put  $\mathcal{P}''_n = \{\{x\} : \{x\} \in \mathcal{B}_n\}$  for each  $n \in \mathbb{N}$ . Then  $\mathcal{P}''_n$  is discrete for each  $n \in \mathbb{N}$ . Let  $\mathcal{P}' = \bigcup_{n \in \mathbb{N}} (\mathcal{P}'_n \cup \mathcal{P}''_n)$ . It is easy to see that  $\mathcal{P}'$  is a  $\sigma$ -locally finite  $\pi$ -base for  $X$ . Indeed, for each nonempty open subset  $O$  of  $X$ , if  $O \cap I(X) \neq \emptyset$ , then we choose a point  $x \in O \cap I(X)$  and therefore,  $\{x\} \in \mathcal{B}$  and  $\{x\} \subset O$ ; if  $O \cap I(X) = \emptyset$ , then there is a  $B \in \mathcal{B}$  with  $B \subset O$  since  $\mathcal{B}$  is a  $\pi_*$ -base, and therefore,  $\emptyset \neq \text{int}(B) \cap X(n) \subset O$  by the Claim.  $\square$

**Corollary 2.4.** *A space  $X$  is  $\pi$ -metrizable if and only if  $X$  has a  $\sigma$ -HCP  $\pi_*$ -base  $\mathcal{P}$  such that, for every  $P \in \mathcal{P}$ ,  $P$  is a regular closed set of  $X$ .*

In [10], D. Stover has proved that open perfect or irreducible perfect maps preserve  $\pi$ -metrizability. However, we have the following Theorem 2.5, Corollaries 2.6 and 2.7, which give an affirmative answer for Question 1.2 and also improve some results in [10].

**Theorem 2.5.** *Quasi-open and closed maps preserve  $\pi$ -metrizability.*

*Proof.* Let  $f : X \rightarrow Y$  be an quasi-open and closed map, where  $X$  is a  $\pi$ -metrizable space. It follows from Theorem 2.3 that  $X$  has a  $\sigma$ -HCP  $\pi$ -base  $\mathcal{P}$ . Since closed maps preserve HCP collections,  $f(\mathcal{P})$  is a  $\sigma$ -HCP collection of subsets of  $Y$ . Since  $f$  is a quasi-open map,  $\{\text{int}f(P) : P \in \mathcal{P}\}$  is a  $\pi$ -base for  $X$ . Hence  $Y$  is a  $\pi$ -metrizable space by Theorem 2.3.  $\square$

**Corollary 2.6.** *Open and closed maps preserve  $\pi$ -metrizability.*

**Corollary 2.7.** *Irreducible closed maps preserve  $\pi$ -metrizability.*

*Proof.* It follows from the definition of the irreducible closed mappings that an irreducible closed map is quasi-open. Therefore, irreducible closed maps preserve  $\pi$ -metrizability by Theorem 2.5.  $\square$

However, perfect maps don't preserve  $\pi$ -metrizability, see Example 2.14.

A topological property  $\mathcal{P}$  satisfies *hereditarily closure-preserving regular closed sum theorems* if a topological space  $X$  has a hereditarily closure-preserving regular closed covering  $\{F_\alpha\}_{\alpha \in A}$  such that  $F_\alpha$  has topological property  $\mathcal{P}$  for every  $\alpha \in A$ , then  $X$  has topological property  $\mathcal{P}$ .

**Lemma 2.8.** *Suppose the topological property  $\mathcal{P}$  satisfies the following two conditions:*

- (1)  $\mathcal{P}$  is preserved by topological sums;
- (2)  $\mathcal{P}$  is preserved by quasi-open and closed maps,

*then  $\mathcal{P}$  satisfies hereditarily closure-preserving regular closed sum theorem.*

*Proof.* Let  $\{F_\alpha\}_{\alpha \in A}$  be a hereditarily closure-preserving regular closed covering for a space  $X$ , where  $F_\alpha$  has topological property  $\mathcal{P}$  for every  $\alpha \in A$ . For every  $\alpha \in A$ , let  $F'_\alpha$  denote a copy of  $F_\alpha$  and let  $f_\alpha$  be this homeomorphism. Put  $X^*$  be the disjoint topological sum of  $F'_\alpha$ , and define a map  $f$  from  $X^*$  onto  $X$  as follows: for every  $x \in X^*$ , if  $x \in F'_\alpha$ , then  $f(x) = f_\alpha(x)$ .

Obviously,  $f$  is a map. It follows from (1) that  $X^*$  has topological property  $\mathcal{P}$ . It is easy to see that  $f$  is a closed map since  $\{F_\alpha\}_{\alpha \in A}$  is HCP. Now we only need to

show that  $f$  is a quasi-open map. Since  $F_\alpha$  is a regular closed set, there is an open subset  $U_\alpha$  of  $X$  such that  $F_\alpha = \overline{U_\alpha}$ . Clearly, it is sufficient to show that  $\text{int}f(E) \neq \emptyset$  for each non-empty open subset  $E$  in  $F'_\alpha$ . Since  $f_\alpha : F'_\alpha \rightarrow \overline{U_\alpha}$  is a homeomorphism map,  $f_\alpha(E)$  is open in  $\overline{U_\alpha}$ , and therefore, there exists an open subset  $U$  in  $X$  such that  $f_\alpha(E) = U \cap \overline{U_\alpha}$ . Choose a point  $x \in f_\alpha(E)$ . Then there is an open subset  $V(x)$  of  $X$  such that  $x \in V(x) \subset U$ . Since  $x \in f_\alpha(E) \subset \overline{U_\alpha}$ ,  $V(x) \cap U_\alpha \neq \emptyset$ . Then  $V(x) \cap U_\alpha \subset f_\alpha(E)$ , and hence  $\text{int}f_\alpha(E) \neq \emptyset$ . Since  $E \subset F'_\alpha$ ,  $f(E) = f_\alpha(E)$ . Then  $f$  is quasi-open. Therefore,  $X$  has topological property  $\mathcal{P}$  by (2).  $\square$

**Theorem 2.9.**  *$\pi$ -metrizability satisfies hereditarily closure-preserving regular closed sum theorems.*

*Proof.* It is easy to prove that  $\pi$ -metrizability is preserved by topological sums. Since  $\pi$ -metrizability is preserved by quasi-open and closed maps,  $\pi$ -metrizability satisfies locally finite regular closed sum theorem by Lemma 2.8.  $\square$

It is well known that a space  $X$  is metrizable if and only if  $X$  is paracompact and locally metrizable. However, there exists a  $\pi$ -metrizable space such that  $X$  is non-paracompact. But we have the following Theorem 2.10.

A space  $X$  is called *almost  $\sigma$ -paracompact* if, for each open covering  $\mathcal{U}$  of  $X$ , there is a  $\sigma$ -locally finite open collection  $\mathcal{V}$  such that  $\mathcal{V}$  refines  $\mathcal{U}$  and  $\cup \mathcal{V}$  is dense in  $X$ . Obviously, paracompact or  $\pi$ -metrizable spaces are almost  $\sigma$ -paracompact.

**Theorem 2.10.** *A space  $X$  is  $\pi$ -metrizable if and only if  $X$  is almost  $\sigma$ -paracompact and locally  $\pi$ -metrizable.*

*Proof.* Obviously, we only need to show the sufficiency.

Let  $X$  be almost  $\sigma$ -paracompact and locally  $\pi$ -metrizable. For each  $x \in X$ , there exists an open neighborhood  $V_x$  of  $x$  such that  $V_x$  is  $\pi$ -metrizable. Then  $\{V_x : x \in X\}$  is an open covering for  $X$ . Since  $X$  is almost  $\sigma$ -paracompact, there exists a  $\sigma$ -locally finite open collection  $\mathcal{V}$  refining  $\{V_x : x \in X\}$  and  $\cup \mathcal{V}$  is dense in  $X$ . We denote  $\mathcal{V}$  by  $\mathcal{V} = \bigcup_{m \in \mathbb{N}} \mathcal{V}_m$ . By the regularity, we can assume that  $\overline{\mathcal{V}} = \{\overline{V} : V \in \mathcal{V}\}$  refines  $\{V_x : x \in X\}$ . Obviously,  $\overline{\mathcal{V}}$  is  $\sigma$ -locally finite. Fix an  $m \in \mathbb{N}$ . For each  $\overline{V} \in \overline{\mathcal{V}}_m$ , since  $\pi$ -metrizability is preserved by the closure of open subspaces,  $\overline{V}$  is  $\pi$ -metrizable, and therefore, let  $\mathcal{P}(\overline{V}) = \bigcup_{n \in \mathbb{N}} \mathcal{P}_{mn}(\overline{V})$  be a  $\sigma$ -discrete  $\pi$ -base for  $\overline{V}$ , where  $\mathcal{P}_n(\overline{V})$  is discrete in  $\overline{V}$  for each  $n \in \mathbb{N}$ . In fact, for each  $\overline{V} \in \overline{\mathcal{V}}$  and  $W \in \mathcal{P}(\overline{V})$ , we can also assume that  $W \subset V$ . Put  $\mathcal{P}_{mn} = \bigcup_{\overline{V} \in \overline{\mathcal{V}}} \mathcal{P}_{mn}(\overline{V})$  and  $\mathcal{P} = \bigcup_{m,n \in \mathbb{N}} \mathcal{P}_{mn}$ . Then  $\mathcal{P}$  is a  $\sigma$ -locally finite  $\pi$ -base for  $X$ . Firstly,  $\mathcal{P}$  is a  $\pi$ -base for  $X$ . In fact, let  $U$  be a nonempty open subset for  $X$ . Since  $\cup \mathcal{V}$  is dense in  $X$ , there is an  $V \in \mathcal{V}$  such that  $U \cap V \neq \emptyset$ . It follows from  $W \subset V$  for each  $W \in \mathcal{P}(\overline{V})$  that there exists a  $W \in \mathcal{P}(\overline{V})$  such that  $W \subset U \cap V$ . Now, we show that  $\mathcal{P}_{mn}$  is locally finite for each  $m, n \in \mathbb{N}$ . For each  $x \in X$ , since  $\overline{\mathcal{V}}_m$  is locally finite, there exists an open neighborhood  $U(x)$  of  $x$  such that  $U(x)$  intersects only finitely many elements of  $\overline{\mathcal{V}}_m$ . We denote those finitely many elements by  $\overline{V}_1, \dots, \overline{V}_k$ . Then we need only to find an open neighborhood  $G$  of  $x$  such that  $G$  intersects only finitely many elements of  $\bigcup_{i=1}^k \mathcal{P}_{mn}(\overline{V}_i)$ . Clearly,  $\mathcal{P}_{mn}(\overline{V}_i)$  is locally finite in  $X$  for each  $1 \leq i \leq k$ . For each  $1 \leq i \leq k$ , there exists an open subset  $U_i$  with  $x \in U_i$  such that  $U_i$  intersects only finitely many elements of  $\mathcal{P}_{mn}(\overline{V}_i)$ . Let  $G = U(x) \cap (\bigcap_{i=1}^k U_i)$ . Clearly,  $G$  is an open neighborhood of  $x$  and intersects only finitely many elements of  $\bigcup_{i=1}^k \mathcal{P}_{mn}(\overline{V}_i)$ .  $\square$

**Remark** (1) We can not omit the condition “ $X$  is almost  $\sigma$ -paracompact” in Theorem 2.10. Indeed, Isbell-Mrówka space  $\psi(D)$  [2] is locally  $\pi$ -metrizable and non- $\pi$ -metrizable, where  $D$  is a discrete space with  $|D| = \aleph_1$ . However, it is easy to see that  $\psi(D)$  is not an almost  $\sigma$ -paracompact space.

(2) We can not replace “ $X$  is almost  $\sigma$ -paracompact” by “ $X$  is almost paracompact” in Theorem 2.10, where a space is called *almost paracompact* [9] if, for each open covering  $\mathcal{U}$ , there is a locally finite open collection  $\mathcal{V}$  such that  $\mathcal{V}$  refines  $\mathcal{U}$  and  $\cup \mathcal{V}$  is dense in  $X$ . In fact, Isbell-Mrówka space  $\psi(\mathbb{N})$  [2] is a  $\pi$ -metrizable space and non-almost paracompact.

Next, we discuss the second-countable  $\pi$ -metrizable spaces.

It is clear that second-countable  $\pi$ -metrizability is preserved by open subspaces, closures of open subspaces, and dense subspaces. As countability, let  $X$  be a  $\pi$ -metrizable space. Then  $X$  is a second-countable  $\pi$ -metrizable space if  $X$  satisfies one of the following conditions:

- (1)  $X$  is separable;
- (2)  $X$  is Lindelöf;
- (3)  $X$  is pseudocompact.

**Remark** It is well known, for a metrizable space  $X$ , that  $X$  is separable if and only if  $X$  is Lindelöf. However, there is a separable and  $\pi$ -metrizable space, which is not a Lindelöf space, for example, Isbell-Mrówka space  $\psi(\mathbb{N})$  [2].

The following result is easy to see.

**Proposition 2.11.** Second-countable  $\pi$ -metrizability is preserved by quasi-open maps.

However, there exists a non- $\pi$ -metrizable space, which is the image of a second-countable  $\pi$ -metrizable space under a closed and at most two-to-one map, see Example 2.14.

**Theorem 2.12.** A space  $Y$  is the image of a second-countable  $\pi$ -metrizable space  $X$  under a closed and at most two-to-one map if and only if  $Y$  is separable.

*Proof.* Necessity. Since a second-countable  $\pi$ -metrizable space is separable,  $Y$  is separable.

Sufficiency. If  $Y$  is finite, it is obvious. Hence we can assume that  $Y$  is infinite. Let  $\{d_n : n \in \mathbb{N}\}$  be a countable dense subset for  $Y$ , where  $d_n \neq d_m$  for distinct  $n, m \in \mathbb{N}$ . Let  $X = \{(n, d_n) : n \in \mathbb{N}\} \cup (\{p\} \times Y)$  and endow  $X$  with the subspace topology of  $\mathbb{N}_* \times Y$ , where  $\mathbb{N}_* = \mathbb{N} \cup \{p\}$  is the Alexandroff compactification of  $\mathbb{N}$ .

Claim:  $X$  is second-countable  $\pi$ -metrizable.

Let  $\mathcal{P}_n = \{(n, d_n)\}$  and  $\mathcal{B}_n = \{(p, d_n) : \{d_n\} \in \tau(Y)\}$  for each  $n \in \mathbb{N}$ . Obviously,  $\mathcal{P}_n$  and  $\mathcal{B}_n$  are discrete for each  $n \in \mathbb{N}$ , where  $\mathcal{B}_n = \emptyset$  if  $\{d_n\} \notin \tau(Y)$ . Then  $\bigcup_{n \in \mathbb{N}} (\mathcal{P}_n \cup \mathcal{B}_n)$  is a  $\pi$ -base for  $X$ . Indeed, let  $O$  be a nonempty open subset of  $X$ . Then there exist an  $m \in \mathbb{N}$  and an open subset  $U$  of  $Y$  such that  $O = ((\mathbb{N}_* \setminus \{1, 2, \dots, m-1\}) \times U) \cap X$ . Obviously, we only need to prove that  $O \cap \{(n, d_n) : n \in \mathbb{N}\} \neq \emptyset$  or  $O \cap L \neq \emptyset$ , where  $L = \{(p, d_n) : \{d_n\} \in \tau(Y)\}$ . If  $O \cap (\{p\} \times Y) = \emptyset$ , then it is obvious. Therefore, we can assume that  $O \cap (\{p\} \times Y) \neq \emptyset$ . Suppose that  $O \cap \{(n, d_n) : n \in \mathbb{N}\} = \emptyset$ . Then  $O \subset \{p\} \times Y$ . Since  $U$  is open in  $Y$ , there exists an  $n \in \mathbb{N}$  such that  $d_n \in U$ . Assume that  $O \cap L = \emptyset$ . Then  $(n, d_n) \in O$  if  $n \geq m$ , this is a contradiction. Hence  $n < m$ . Since  $U \setminus \{d_1, d_2, \dots, d_{m-1}\} \neq \emptyset$ , there is an  $n_0 \geq m$  such that  $d_{n_0} \in U$ . Therefore,  $(n_0, d_{n_0}) \in O$ , this is a contradiction. Hence  $O \cap L \neq \emptyset$ . Then there exists a  $k \in \mathbb{N}$  such that  $(p, d_k) \in \mathcal{B}_k$  and  $(p, d_k) \in O \cap L$ .

Let  $f : X \rightarrow Y$  be the natural projection map. Since  $\mathbb{N}_*$  is compact, the projection of  $\mathbb{N}_* \times Y$  onto  $Y$  is a closed map. It follows from  $X$  is a closed subspace of  $\mathbb{N}_* \times Y$  that  $f$  is a closed map. Obviously, for each  $y \in Y$ ,  $f^{-1}(y)$  is at most two points set. Hence  $f$  is a closed and at most two-to-one map.  $\square$

**Corollary 2.13.** *A space  $Y$  is the image of a second-countable  $\pi$ -metrizable space  $X$  if and only if  $Y$  is separable.*

*Example 2.14.* There exists a regular and separable space  $X$ , which is not a  $\pi$ -metrizable space. Therefore, closed and at most two-to-one maps don't preserve  $\pi$ -metrizability by Theorem 2.12.

*Proof.* Suppose that  $\mathbb{I} = [0, 1]$  is the closed unit interval with a subspace of the usual topology  $\mathbb{R}$ , and  $X = \mathbb{I}^{\mathbb{I}}$  with the product topology. Then  $X$  is a regular and separable space. However,  $X$  is not a  $\pi$ -metrizable space by [10, Theorem 3.11].  $\square$

### 3. STRONGLY $\pi$ -METRIZABLE SPACES

**Definition 3.1.** Let  $\mathcal{P}$  be a collection of open subsets of  $X$ .  $\mathcal{P}$  is called a *strong  $\pi$ -base* [1] if  $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$  and, for each  $x \in X$ ,  $\mathcal{P}_x$  is a strong  $\pi$ -base at point  $x$ , that is,  $\mathcal{P}_x$  is a  $\pi$ -base at point  $x$  and every open neighborhood of  $x$  contains all but finitely many elements of  $\mathcal{P}_x$ .

$X$  is called *strongly  $\pi$ -metrizable* if  $X$  has a  $\sigma$ -discrete strong  $\pi$ -base.  $X$  is called *second-countable strongly  $\pi$ -metrizable* if  $X$  has a countably strong  $\pi$ -base.

It is obvious that every metrizable space is strongly  $\pi$ -metrizable, and every strongly  $\pi$ -metrizable space is  $\pi$ -metrizable. The implications of the converses are not true.

(1) Isbell-Mrówka space  $\psi(\mathbb{N})$  [2] is a strongly  $\pi$ -metrizable space, but it is not a metrizable space;

(2) Let  $K$  be a discrete space with  $|K| = \aleph_1$ .  $K^{\aleph_1}$  is  $\pi$ -metrizable by [10], and however,  $K^{\aleph_1}$  is a non-strongly  $\pi$ -metrizable space by the following Theorem 3.18.

Clearly, if  $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$  is a strong  $\pi$ -base for  $X$ , then every infinite subfamily of  $\mathcal{P}_x$  is a strong  $\pi$ -base at point  $x$ . Therefore, we have the following result.

**Proposition 3.2.** If  $X$  has a strong  $\pi$ -base, then every point of  $X$  has a countably strong  $\pi$ -base.

In [10], D. Stover given a non-metrizable topological group, which is  $\pi$ -metrizable. However, we have the following result by Theorem 3.2.

**Theorem 3.3.** *If  $X$  is a topological group with a strong  $\pi$ -base, then  $X$  is metrizable.*

*Proof.* Obviously,  $X$  has a countable  $\pi$ -character by Proposition 3.2. Then  $X$  is metrizable by [4, Theorems 3.6 and 1.8].  $\square$

**Theorem 3.4.** *For a topological space  $X$ , the following are equivalent:*

- (1)  $X$  is a strongly  $\pi$ -metrizable space;
- (2)  $X$  has a  $\sigma$ -HCP strong  $\pi$ -base;
- (3)  $X$  has a  $\sigma$ -locally finite strong  $\pi$ -base.

*Proof.* (1) $\Rightarrow$ (2). It is obvious. From [10, Lemma 2.1], it is easy to see that (3) $\Rightarrow$ (1).

It is easy to see that (2) $\Rightarrow$ (3) by the proof of (2) $\Rightarrow$ (3) in Theorem 2.3.  $\square$

It is obvious that strongly  $\pi$ -metrizability is preserved by open subspaces or dense subspaces. However, we have the following questions.

**Question 3.5.** *Is strongly  $\pi$ -metrizability preserved by the closures of open subspaces?*

**Question 3.6.** *Let  $X$  be a paracompact space. If  $X$  is locally strongly  $\pi$ -metrizable, then is  $X$  strongly  $\pi$ -metrizable?*

**Theorem 3.7.** *Quasi-open and closed maps preserve strongly  $\pi$ -metrizability.*

*Proof.* Let  $f : X \rightarrow Y$  be an open and closed map, where  $X$  is a strongly  $\pi$ -metrizable space. It follows from Theorem 3.4 that  $X$  has a  $\sigma$ -HCP strong  $\pi$ -base  $\mathcal{P}$ . Since closed maps preserve HCP collections,  $f(\mathcal{P})$  is a  $\sigma$ -HCP collection of subsets of  $Y$ . Since  $f$  is a quasi-open map,  $\{\text{int} f(P) : P \in \mathcal{P}\}$  is a  $\sigma$ -HCP strong  $\pi$ -base of  $Y$ . In fact, for each  $y \in Y$ , choose a fixed point  $x_y \in f^{-1}(y)$ . Then  $\{\text{int} f(P) : P \in \mathcal{P}_{x_y}\}$  is a strong  $\pi$ -base at point  $y$ . Hence  $Y$  is a strongly  $\pi$ -metrizable space by Theorem 3.4.  $\square$

**Corollary 3.8.** *Open and closed maps preserve strongly  $\pi$ -metrizability.*

**Corollary 3.9.** *Irreducible closed maps preserve strongly  $\pi$ -metrizability.*

*Proof.* It is easy to see by Theorem 3.7 and the proof of Corollary 2.7.  $\square$

*Example 3.10.* There exists a non-strongly  $\pi$ -metrizable space  $X$ , which is the inverse image of a strongly  $\pi$ -metrizable space under a perfect map.

*Proof.* Let  $D$  be an uncountable set and endow  $D$  with a discrete topology. Let  $Z$  be the Alexandroff compactification of  $D$ , that is,  $Z = D \cup \{z\}$ . Let  $X = \psi(\mathbb{N}) \times Z$  be the product topology, where  $\psi(\mathbb{N})$  is Isbell-Mrówka space. Then the projection  $\pi_1 : X \rightarrow \psi(\mathbb{N})$  is a perfect map. However,  $X$  is a non-strongly  $\pi$ -metrizable space. Suppose not, then the point  $x = (1, z) \in X$  has a countable strong  $\pi$ -base  $\mathcal{P}_x$  by Proposition 3.2. Since every open neighborhood of  $z$  in  $Z$  has the form  $Z - A$  with  $A$  a finite subset of  $D$ , there exists a countable subset  $L \subset D$  such that  $(D - L) \subset \pi_2(P)$  for each  $P \in \mathcal{P}_x$ . Choose a point  $y \in D - L$ . Then  $\{1\} \times (Z - \{y\})$  is an open neighborhood of  $(1, z)$ . But  $P \not\subset \{1\} \times (Z - \{y\})$  for each  $P \in \mathcal{P}_x$ , this is a contradiction.  $\square$

**Question 3.11.** *Do irreducible perfect maps inversely preserve strongly  $\pi$ -metrizability?*

**Theorem 3.12.** *Let  $Y$  be a Fréchet space. Then  $Y$  is the image of a strongly  $\pi$ -metrizable space  $X$  under a perfect map if and only if  $Y$  is strongly  $d$ -separable.*

*Proof.* Necessity. It is obvious.

Sufficiency. Let  $\bigcup_{n \in \mathbb{N}} D_n$  be a dense subset for  $Y$ , where  $D_n$  is a closed and discrete subspace of  $Y$  for each  $n \in \mathbb{N}$ . Put  $E_n = \bigcup_{i=1}^n D_i$  for each  $n \in \mathbb{N}$ . Obviously, for each  $n \in \mathbb{N}$ ,  $E_n$  is a closed and discrete subspace of  $Y$ .

By the same notations in Theorem 2.12. Let  $X = (\bigcup \{\{n\} \times E_n : n \in \mathbb{N}\}) \cup (\{p\} \times Y)$ . Then  $X$  is a strongly  $\pi$ -metrizable space. Indeed, let  $\mathcal{B}_n = \{(n, d) : d \in E_n\}$  for each  $n \in \mathbb{N}$ . Obviously,  $\mathcal{B}_n$  is discrete for each  $n \in \mathbb{N}$ . Then  $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n$  is a strong  $\pi$ -base for  $X$ .

(i) If  $x = (n, d) \in \{n\} \times E_n$  for some  $n \in \mathbb{N}$ , then let  $\mathcal{B}_x = \{(n, d)\}$ , and therefore,  $\mathcal{B}_x$  is a strong  $\pi$ -base at point  $x$ .

(ii) If  $x = (p, d) \in \{p\} \times \bigcup_{n \in \mathbb{N}} E_n$ , then there exists an  $m \in \mathbb{N}$  such that  $d \in E_m$ . We let  $\mathcal{B}_x = \{(i, d) : i \geq m\} \subset \mathcal{B}$ . Then  $\mathcal{B}_x$  is a strong  $\pi$ -base at point  $x$ .

(iii) If  $x = (p, d) \in \{p\} \times (Y \setminus \bigcup_{n \in \mathbb{N}} E_n)$ , then  $d \in \overline{\bigcup_{n \in \mathbb{N}} E_n}$ . Since  $Y$  is Fréchet, there exists a sequence  $\{d_n\}_{n=1}^\infty$  in  $\bigcup_{n \in \mathbb{N}} E_n$  such that  $d_n \rightarrow d$  as  $n \rightarrow \infty$ . By the induction on  $\mathbb{N}$ , we can define an increasing sequence  $\{m_{d_n}\}_{n=1}^\infty$  in  $\mathbb{N}$  such that, for each  $n \in \mathbb{N}$ ,  $m_{d_n} > n$ , and  $d_n \in E_{m_{d_n}}$ . Let  $\mathcal{B}_x = \{(m_{d_n}, d_n) : n \in \mathbb{N}\}$ . Then  $\mathcal{B}_x$  is a strong  $\pi$ -base at point  $x$ . In fact, let  $O$  be an open neighborhood at point  $x$ . Then there exist an

$\mathbb{N}_m = \mathbb{N}_* \setminus \{1, 2, \dots, m-1\}$  and an open neighborhood  $U$  at point  $d$  in  $Y$  such that  $(\mathbb{N}_m \times U) \cap X \subset O$ . Since  $d_n \rightarrow d$ , there is a  $l \in \mathbb{N}$  such that  $\{d_n : n \geq l\} \subset U$ . Put  $k = \max\{l, m\}$ . Then, for each  $n \geq k$ , we have  $(m_{d_n}, d_n) \in (\mathbb{N}_m \times U) \cap X$ .

Let  $f : X \rightarrow Y$  be the natural projection map. Since  $\mathbb{N}_*$  is compact, the projection of  $\mathbb{N}_* \times Y$  onto  $Y$  is a closed map. It follows from  $X$  is a closed subspace of  $\mathbb{N}_* \times Y$  that  $f$  is a closed map. For each  $y \in Y$ , since  $f^{-1}(y)$  is homeomorphic to a subspace of  $\mathbb{N}_*$  containing the limit point  $p$ ,  $f^{-1}(y)$  is compact. Hence  $f$  is a perfect map.  $\square$

**Corollary 3.13.** *Let  $Y$  be a Fréchet space. Then  $Y$  is the image of a strongly  $\pi$ -metrizable space  $X$  under a closed map if and only if  $Y$  is strongly  $d$ -separable.*

**Theorem 3.14.** *Let  $Y$  be a Fréchet space. Then  $Y$  is the image of a second-countable strongly  $\pi$ -metrizable space  $X$  under a closed and at most two-to-one map if and only if  $Y$  is separable.*

*Proof.* By the same notations in Theorem 2.12. Let  $X = \{(n, d_n) : n \in \mathbb{N}\} \cup (\{p\} \times Y)$ ,  $\mathcal{P}_n = \{(n, d_n)\}$  and  $\mathcal{B}_n = \{(p, d_n) : \{d_n\} \in \tau(Y)\}$  for each  $n \in \mathbb{N}$ . By a similar argument of Theorem 3.12, we can show that  $\bigcup_{n \in \mathbb{N}} (\mathcal{P}_n \cup \mathcal{B}_n)$  is a countably strong  $\pi$ -base for  $X$  and  $Y$  is the image of  $X$  under a closed and at most two-to-one map.  $\square$

*Example 3.15.* There exists a Fréchet,  $\pi$ -metrizable, separable, regular, and non-strongly  $\pi$ -metrizable space  $X$ . Therefore, closed and at most two-to-one maps don't preserve strongly  $\pi$ -metrizability by Theorem 3.14.

*Proof.* Let  $X$  be the sequence fan space  $S_\omega$ , which is obtained from the topological sum of  $\omega$  many copies of the convergent sequence by identifying all the limit points to a point. Then  $X$  is Fréchet,  $\pi$ -metrizable, regular, and separable. Let  $X = \{x_{ni} : i, n \in \mathbb{N}\} \cup \{a\}$ , where  $x_{ni} \rightarrow a$  as  $i \rightarrow \infty$  for each  $n \in \mathbb{N}$ . However,  $X$  is non-strongly  $\pi$ -metrizable. Suppose not, there exists a collection  $\mathcal{P}_a$  of open subsets of  $X$  such that  $\mathcal{P}_a$  is a strong  $\pi$ -base at point  $a$ . By an induction on  $\mathbb{N}$ , we can choose an increasing sequence  $\{n_k\}_k \subset \mathbb{N}$  and a subfamily  $\{P_{n_k} : k \in \mathbb{N}\}$  of  $\mathcal{P}_a$  such that, for each  $k \in \mathbb{N}$ ,  $P_{n_k} \cap \{x_{n_k i} : i \in \mathbb{N}\} \neq \emptyset$  and  $P_{n_{k+1}} \in \mathcal{P}_a \setminus \{P_{n_i} : i \leq k\}$ , where  $P_{n_k} \in \mathcal{P}_a$  for each  $k \in \mathbb{N}$ . Choose a point  $x_{n_k i_{n_k}} \in P_{n_k} \cap \{x_{n_k i} : i \in \mathbb{N}\}$  for each  $k \in \mathbb{N}$ . Then

$$U = \{x_{n_k i} : i > i_{n_k}, k \in \mathbb{N}\} \cup \{x_{ni} : n \in (\mathbb{N} \setminus \{n_k : k \in \mathbb{N}\}), i \in \mathbb{N}\} \cup \{a\}$$

is an open neighborhood of  $a$ . But  $P_{n_k} \not\subset U$  for each  $k \in \mathbb{N}$ , this is a contradiction with  $\mathcal{P}_a$  is a strong  $\pi$ -base at point  $a$ .  $\square$

**Theorem 3.16.** *If  $X_n$  is strongly  $\pi$ -metrizable for each  $n \in \mathbb{N}$ , then  $X = \prod_{n \in \mathbb{N}} X_n$  is strongly  $\pi$ -metrizable.*

*Proof.* It follows from Proposition 3.2 that every point of  $X_n$  has a countably strong  $\pi$ -base for each  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ , let  $\mathcal{P}^n = \bigcup_{x(n) \in X_n} \mathcal{P}_{x(n)}^n$  be a  $\sigma$ -discrete strong  $\pi$ -base for  $X_n$ , where  $\mathcal{P}_{x(n)}^n = \{U_{x(n)}^i : i \in \mathbb{N}\}$ . For every  $x \in X$ , put

$$\mathcal{P}_x = \left\{ \prod_{k=1}^n U_{x(k)}^n \times \prod_{k=n+1}^{\infty} X_k : n \in \mathbb{N} \right\}.$$

Then  $\mathcal{P}_x$  is a strong  $\pi$ -base at point  $x$ . Indeed, for any  $x \in U \in \tau(X)$ , then  $U$  has the form  $U = \prod_{i=1}^m W_i \times \prod_{i=m+1}^{\infty} X_i$ , where  $W_i$  is open in  $X_i$  for each  $1 \leq i \leq m$ . Then for every  $1 \leq i \leq m$ , there is a  $k_i \in \mathbb{N}$  such that  $U_{x(i)}^n \subset W_i$  for every  $n \geq k_i$ . Put  $k_0 = \max\{k_1, \dots, k_m, m\}$ . Therefore, for every  $n > k_0$  and  $1 \leq i \leq m$ ,  $U_{x(i)}^n \subset W_i$ , and hence  $\prod_{k=1}^n U_{x(k)}^n \times \prod_{k=n+1}^{\infty} X_k \subset U$  for every  $n > k_0$ .



Let  $\mathcal{P} = \bigcup_{x \in X} \mathcal{P}_x$ . By the proof of [10, Proposition 3.1], it is easy to see that  $\mathcal{P}$  is  $\sigma$ -locally finite. Hence  $X$  is strongly  $\pi$ -metrizable by Theorem 3.4.  $\square$

**Corollary 3.17.** *If  $X_n$  has a strong  $\pi$ -base for each  $n \in \mathbb{N}$ , then  $X = \prod_{n \in \mathbb{N}} X_n$  also has a strong  $\pi$ -base.*

**Theorem 3.18.** *Let  $\kappa$  be an uncountable cardinal numbers. If  $X_\alpha$  contains at least two points for each  $\alpha < \kappa$ , then the product topology  $X = \prod_{\alpha < \kappa} X_\alpha$  does not have a strong  $\pi$ -base at any point of  $X$ . In particular,  $X$  is non-strongly  $\pi$ -metrizable.*

*Proof.* Suppose not; there is a point  $x \in X$  such that the point  $x$  has a countably strong  $\pi$ -base  $\mathcal{P}_x$ . Then there exists a  $\beta < \kappa$  such that  $\pi_\beta(P) = X_\beta$  for each  $P \in \mathcal{P}_x$ . Since  $X_\beta$  is at least two points set, we choose a point  $y \in X_\beta \setminus \{\pi_\beta(x)\}$ . Then  $(X_\beta \setminus \{y\}) \times \prod_{\alpha \in (\kappa - \{\beta\})} X_\alpha$  is an open neighborhood of  $x$ . However,  $P \notin (X_\beta \setminus \{y\}) \times \prod_{\alpha \in (\kappa - \{\beta\})} X_\alpha$  for each  $P \in \mathcal{P}_x$ , this is a contradiction.  $\square$

**Question 3.19.** *Is it true that for any non-strongly  $\pi$ -metrizable spaces  $X$  and  $Y$ , we have that  $X \times Y$  is also non-strongly  $\pi$ -metrizable?*

**Question 3.20.** *Does there exist a non-strongly  $\pi$ -metrizable space  $X$  such that  $X^n$  is strongly  $\pi$ -metrizable for some  $n \in \mathbb{N}$ ?*

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